JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 51, 56–67 (1975)

# Some Results on the Precompactness of Orbits of Dynamical Systems*

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## 1. INTRODUCTION

The concept of the positive limit set of a positive orbit is basic to the theory of dynamical systems in Banach spaces. If a positive orbit belongs to a compact subset of the space the corresponding positive limit set will be nonempty, invariant and connected. In this case it is often possible to draw strong conclusions concerning the asymptotic behavior of motions through the application of the well known invariance principle [1–6].

In applications it is usually much easier to show that a positive orbit belongs to a bounded set than to prove that it lies in a compact set. In this paper we present certain results which may be useful in dealing with the question of compactness and, hence, which might be attractive in the application of the Direct Method of Liapunov and its various extensions to the stability analysis of general dynamical systems. These results formalize and extend a variety of devices previously used to obtain information on compactness [6–9].

Let $\mathscr{C}$ be a subset of a Banach space $\mathscr{X}$ and let $\{u(t, \cdot)\}_{t\geqslant 0}$ be a strongly continuous semigroup of continuous operators on $\mathscr{C}$. Thus for $t$, $\tau \geqslant 0$, $\phi \in \mathscr{C}$, we assume that $u(0, \phi) = \phi$, $u(t + \tau, \phi) = u(t, u(\tau, \phi))$, and the map-

* This research was supported by the National Science Foundation under GK 40009 and Office of Naval Research under NONR N00014-67-A-0191-0009.

pings $u(\cdot, \phi): \mathscr{R}^+ \to (\mathscr{C} \subset \mathscr{X})$, $u(t, \cdot): (\mathscr{C} \subset \mathscr{X}) \to (\mathscr{C} \subset \mathscr{X})$, are continuous, where $\mathscr{R}^+$ is the nonnegative real line. In the language of topological dynamics $u(\cdot, \phi)$ is the *motion* through $\phi \in \mathscr{C}$, $u(t, \phi)$ is the *state* at $t \geqslant 0$ resulting from the initial state $\phi$ at $t = 0$, $\gamma^+(\phi) = \bigcup_{t \geqslant 0} u(t, \phi)$ is the *positive orbit* through $\phi$, and $u: \mathscr{R}^+ \times \mathscr{C} \to \mathscr{C}$ is a *dynamical system* on $\mathscr{C} \subset \mathscr{X}$.[1]

Consider the evolution equation

$$\begin{aligned} \frac{d}{dt} v(t) &= Av(t) \quad \text{a.e.} \qquad t \geqslant 0, \\ v(0) &= x \in \mathscr{D}(A), \end{aligned} \tag{1.1}$$

where $A: \mathscr{D}(A) \to \mathscr{X}$, $\mathscr{D}(A) \subset \mathscr{X}$. A mapping $v(\cdot): \mathscr{R}^+ \to \mathscr{X}$ is said to be a *strong solution* if $v(\cdot)$ is absolutely continuous, $v(\cdot)$ is differentiable a.e. on $\mathscr{R}^+$, $v(t) \in \mathscr{D}(A)$ a.e. on $\mathscr{R}^+$, and $v(\cdot)$ satisfies (1.1) [10]. A dynamical system $u$ on $\mathscr{C} \subset \mathscr{X}$ is said to be *generated* by (1.1) if

(i) $\mathscr{D}(A) \subset \mathscr{C} \subset \overline{\mathscr{D}(A)}$,

(ii) $u(\cdot, \phi)$ is a strong solution for each $\phi \in \mathscr{D}(A)$, and

(iii) strong solutions of (1.1) are unique.

Not all dynamical systems are generated by evolution equations, and here we do not assume this property unless it is specifically stated.

A motion $u(\cdot, \phi)$ is said to be *bounded* if $\gamma^+(\phi)$ is a bounded set. Obviously, all motions are bounded if $\mathscr{C}$ is a bounded set. A motion $u(\cdot, \phi)$ is said to be *stable* if given $\epsilon > 0$ there exists $\delta > 0$ such that $\| u(t, x) - u(t, \phi)\|_{\mathscr{X}} < \epsilon$ for all $t \geqslant 0$ and all $x$ in $\mathscr{C}$ with $\| x - \phi \|_{\mathscr{X}} < \delta$. If there exists $x_0 \in \mathscr{C}$ such that $u(t, x_0) \equiv x_0$ (i.e., $\gamma^+(x_0) = \{x_0\}$), $x_0$ is said to be an *equilibrium*. If $u$ has one or more equilibria and all motions are stable, it follows that all motions are bounded. If $u$ is generated by (1.1) there exists an equilibrium $x_0 \in \mathscr{D}(A)$ if and only if $0 \in \mathscr{R}(A)$.

Given a set $\mathscr{S} \subset \mathscr{C}$, $\mathscr{S}$ is said to be *positive invariant* under $u$ if $\gamma^+(\phi) \subset \mathscr{S}$ for every $\phi \in \mathscr{S}$. Clearly a positive orbit, a union of positive orbits, and $\mathscr{C}$ itself are all positive invariant sets. If $\mathscr{S}$ is positive invariant the restriction of $u$ to $\mathscr{S}$ is also a dynamical system on $\mathscr{S} \subset \mathscr{X}$.

Finally, a continuous functional $V: (\mathscr{C} \subset \mathscr{X}) \to \mathscr{R}$ is said to be a *Liapunov functional* if $\dot{V}(x) \leqslant 0$ for all $x \in \mathscr{C}$, where

$$\dot{V}(x) = \limsup_{h \downarrow 0} \frac{1}{h} (V(u(h, x)) - V(x)), \qquad x \in \mathscr{C}.$$

[1] Stronger continuity assumptions can be made when defining a dynamical system [2, 3].

## 2. Homeomorphic State Transformations

For our purposes it is important to describe a class of state transformations under which a dynamical system remains a dynamical system. The following proposition provides a sufficient condition.

Proposition 2.1. *Let $u: \mathscr{R}^+ \times \mathscr{C} \to \mathscr{C}$ be a dynamical system on $\mathscr{C} \subset \mathscr{X}$ where $\mathscr{X}$ is a Banach space. Let $\mathscr{Y}$ be a Banach space and $P: (\mathscr{C} \subset \mathscr{X}) \to (\hat{\mathscr{C}} \subset \mathscr{Y})$ be a homeomorphism. Then the mapping $\hat{u}: \mathscr{R}^+ \times \hat{\mathscr{C}} \to \hat{\mathscr{C}}$ defined by*

$$\hat{u}(t, y) = Pu(t, P^{-1}y), \qquad t \geqslant 0, \quad y \in \hat{\mathscr{C}},$$

*is a dynamical system on $\hat{\mathscr{C}} \subset \mathscr{Y}$. Moreover, if $u$ is generated by* (1.1) *and $P$ has a linear extension defined on all of $\mathscr{X}$, then $\hat{u}$ is generated by*

$$\frac{d}{dt}\hat{u}(t, y) = \hat{A}\hat{u}(t, y) \quad \text{a.e.} \qquad t \geqslant 0, \quad y \in \mathscr{D}(\hat{A}) \subset \mathscr{Y},$$

*where*

$$\mathscr{D}(\hat{A}) = P\mathscr{D}(A), \qquad \hat{A} = PAP^{-1}.$$

*Proof.* By the definition of $\hat{u}$ it is clear that for $t \geqslant 0$, $\psi \in \hat{\mathscr{C}}$, the mappings

$$\hat{u}(\cdot, \psi): \mathscr{R}^+ \to (\hat{\mathscr{C}} \subset \mathscr{Y}) \qquad \text{and} \qquad \hat{u}(t, \cdot): (\hat{\mathscr{C}} \subset \mathscr{Y}) \to (\hat{\mathscr{C}} \subset \mathscr{Y})$$

are continuous and $\hat{u}(0, \psi) = \psi$. In addition, for $t, \tau \geqslant 0$, $\psi \in \hat{\mathscr{C}}$,

$$\begin{aligned} \hat{u}(t + \tau, \psi) &= Pu(t, u(\tau, P^{-1}\psi)) \\ &= Pu(t, P^{-1}\hat{u}(\tau, \psi)) = \hat{u}(t, \hat{u}(\tau, \psi)). \end{aligned}$$

Hence $\hat{u}$ is a dynamical system on $\hat{\mathscr{C}} \subset \mathscr{Y}$.

If $u$ is generated by (1.1) and $P$ has a linear extension on $\mathscr{X}$, which we also denote by $P$, then for $y \in \mathscr{Y}$ such that $P^{-1}y \in \mathscr{D}(A)$ we have

$$\begin{aligned} \frac{d}{dt}\hat{u}(t, y) &= P\frac{d}{dt}u(t, P^{-1}y) \\ &= PAu(t, P^{-1}y) = PAP^{-1}\hat{u}(t, y) \quad \text{a.e.} \qquad t \geqslant 0. \end{aligned}$$

Defining $\mathscr{D}(\hat{A}) = P\mathscr{D}(A)$, $\hat{A} = PAP^{-1}$, we have

$$\hat{u}(t, y) = \hat{A}\hat{u}(t, y) \quad \text{a.e.} \qquad t \geqslant 0, \quad y \in \mathscr{D}(\hat{A}),$$

and

$$\mathscr{D}(\hat{A}) \subset \hat{\mathscr{C}} \subset \overline{\mathscr{D}(\hat{A})} \qquad \text{since} \qquad \mathscr{D}(A) \subset \mathscr{C} \subset \overline{\mathscr{D}(A)}.$$

Several remarks seem appropriate. First of all, it is clear that a set $\mathscr{S}$ is positive invariant under $u$ if and only if $\hat{\mathscr{S}} = P\mathscr{S}$ is positive invariant under $\hat{u}$. Secondly, $V(x)$ is a Liapunov functional for $u$ if and only if $\hat{V}(y) = V(P^{-1}y)$ is a Liapunov functional for $\hat{u}$, since $\dot{\hat{V}}(y) = \dot{V}(P^{-1}y)$. It is also clear that if $P$ is a uniform homeomorphism (for example, if $P$ has a linear extension defined on $\mathscr{X}$) the motion $u(\cdot, \phi)$ is bounded (or stable) if and only if the motion $\hat{u}(\cdot, P\phi)$ is bounded (or stable).

It is often of interest to determine if a given dynamical system $u$ on a set $\mathscr{C} \subset \mathscr{X}$, $\mathscr{X}$ a Banach space, has some restriction which is itself a dynamical system on a set $\hat{\mathscr{C}}$ of another Banach space $\mathscr{Y}$ such that $\mathscr{Y} \subset \mathscr{X}$, $\hat{\mathscr{C}} \subset \mathscr{C}$ [3]. The following proposition provides a sufficient condition for such a result.

PROPOSITION 2.2. *Let $u: \mathscr{R}^+ \times \mathscr{C} \to \mathscr{C}$ be a dynamical system on $\mathscr{C} \subset \mathscr{X}$ where $\mathscr{X}$ is a Banach space. Let $\mathscr{Y}$ be a Banach space with $\mathscr{Y} \subset \mathscr{X}$ and the norm on $\mathscr{Y}$ at least as strong as the norm on $\mathscr{X}$. Let $P: (\mathscr{C} \subset \mathscr{X}) \to (\hat{\mathscr{C}} \subset \mathscr{Y})$ be a homeomorphism such that $\hat{\mathscr{C}} \subset \mathscr{C}$ and*

$$Pu(t, \phi) = u(t, P\phi) \qquad \text{for } t \geqslant 0, \quad \phi \in \mathscr{C}. \tag{2.1}$$

*Then $\hat{\mathscr{C}} \subset \mathscr{X}$ is positive invariant under $u$ and*

$$u(t, y) = \hat{u}(t, y) \qquad \text{for } t \geqslant 0, \quad y \in \hat{\mathscr{C}},$$

*where $\hat{u}$ is the dynamical system on $\hat{\mathscr{C}} \subset \mathscr{Y}$ defined by $\hat{u}(t, y) = Pu(t, P^{-1}y)$, $y \in \hat{\mathscr{C}} \subset \mathscr{Y}$. Moreover, if $u$ is generated by* (1.1), *$P\mathscr{D}(A) \subset \mathscr{D}(A)$, and $P$ has a linear extension defined on all of $\mathscr{X}$, then* (2.1) *is equivalent to the condition*

$$PAx = APx \qquad \text{for } x \in \mathscr{D}(A). \tag{2.2}$$

*Proof.* From the definition of $\hat{u}$, Proposition 2.1 assures us that $\hat{u}$ is a dynamical system on $\hat{\mathscr{C}} \subset \mathscr{Y}$; but then (2.1) implies that $\hat{u}(t, y) = u(t, y)$ for $t \geqslant 0$, $y \in \hat{\mathscr{C}}$. Thus $\hat{\mathscr{C}}$, considered as a subset of $\mathscr{X}$, is positive invariant under $u$.

If $u$ is generated by (1.1), $P\mathscr{D}(A) \subset \mathscr{D}(A)$, and $P$ has a linear extension defined on all of $\mathscr{X}$, then (2.1) implies that

$$P \frac{d}{dt} u(t, \phi) = \frac{d}{dt} u(t, P\phi) \quad \text{a.e.} \qquad t \geqslant 0, \quad \phi \in \mathscr{D}(A).$$

It then follows from (1.1) that

$$\begin{aligned} PAu(t, \phi) &= Au(t, P\phi) \\ &= APu(t, \phi) \quad \text{a.e.} \qquad t \geqslant 0, \quad \phi \in \mathscr{D}(A), \end{aligned}$$

and since $u(0, \phi) = \phi$, this implies that $PA\phi = AP\phi$ for all $\phi \in \mathscr{D}(A)$. On the other hand, (2.2) implies that

$$\frac{d}{dt} Pu(t, \phi) = PAu(t, \phi)$$
$$= APu(t, \phi) \quad \text{a.e.} \qquad t \geqslant 0, \quad \phi \in \mathscr{D}(A).$$

However, $Pu(0, \phi) = P\phi = u(0, P\phi)$ and (1.1) implies

$$\frac{d}{dt} u(t, P\phi) = Au(t, P\phi) \quad \text{a.e.} \qquad t \geqslant 0, \quad \phi \in \mathscr{D}(A),$$

since $\phi \in \mathscr{D}(A)$ implies $P\phi \in \mathscr{D}(A)$. By assumption, solutions of (1.1) are unique and continuous in time. Thus we conclude that $Pu(t, \phi) = u(t, P\phi)$ for all $t \geqslant 0$ when $\phi \in \mathscr{D}(A)$. Furthermore, since the norm on $\mathscr{Y}$ is at least as strong as the norm on $\mathscr{X}$, $Pu(t, \cdot)$ and $u(t, P\cdot)$ are continuous when considered as mappings from $\mathscr{X}$ into $\mathscr{X}$. Since $\mathscr{C} \subset \overline{\mathscr{D}(A)}$, (2.1) follows.

Proposition 2.2 may be viewed as a means of defining a variety of Banach spaces $\mathscr{Y} \subset \mathscr{X}$ such that $u$, restricted to $\hat{\mathscr{C}} \subset \mathscr{C}$, is a dynamical system on $\hat{\mathscr{C}} \subset \mathscr{Y}$ as well as on $\mathscr{C} \subset \mathscr{X}$. If $V(x)$ is a Liapunov functional for $u$ on $\mathscr{C} \subset \mathscr{X}$, then so is $V(Px)$. Moreover $V(P^{-1}x)$ will be a Liapunov functional for the restriction of $u$ to $\hat{\mathscr{C}} \subset \mathscr{Y}$, noting that $V(P^{-1}x)$ is continuous in the topology of $\mathscr{Y}$ but not necessarily in that of $\mathscr{X}$.

Proposition 2.2 represents a generalization of a result of Slemrod [11] for linear strongly continuous semigroups on $\mathscr{C} = \mathscr{X}$ which are generated by (1.1). Slemrod's result follows from Proposition 2.2 upon choosing $P_n = (I - \lambda A)^{-n}$, $n = 1, 2,\ldots$, for sufficiently small real $\lambda > 0$, and noting that various restrictions of $u$ form dynamical systems on each of the spaces

$$\mathscr{Y}_n = \{x \in \mathscr{D}(A^n) \mid \| x \|_{\mathscr{Y}_n} = \|(I - \lambda A)^n x \|_{\mathscr{X}}\},$$

which are Banach spaces since $A$ is linear and closed. Using the Hille–Yosida–Phillips Theorem [12] it is easily shown that $\| x \|_{\mathscr{Y}_n}$ is equivalent to

$$\sum_{i=0}^{n} \| A^i x \|_{\mathscr{X}}$$

for sufficiently small $\lambda > 0$.

Results of this type are of considerable interest in applications of stability theory where the invariance principle is to be used [7–9], for it is then essential to show that positive orbits are precompact. This problem is considered in detail in the following section.

## 3. Precompactness of Positive Orbits

Application of the invariance principle to stability analysis requires the existence of positive limit sets and, hence, precompactness of positive orbits. In applications it is often not too difficult to show that positive orbits are bounded, but this does not imply precompactness unless the Banach space $\mathscr{X}$ is finite dimensional.

Proposition 2.2 suggests one approach to overcoming this difficulty. Suppose that the conditions of that proposition are satisfied, $P$ maps bounded sets into bounded sets, and the natural injection $I: \mathscr{Y} \to \mathscr{X}$ is compact. Then, if $\gamma^+(\phi)$ is bounded for some fixed $\phi \in \mathscr{C}$, it will follow that $\hat{\gamma}^+(P\phi)$ is bounded in $\mathscr{Y}$ and therefore precompact in $\mathscr{X}$. But since

$$\hat{\gamma}^+(P\phi) = \bigcup_{t \geqslant 0} \hat{u}(t, P\phi) = \bigcup_{t \geqslant 0} u(t, P\phi) = \gamma^+(P\phi)$$

it then follows that $\gamma^+(P\phi)$ is precompact in $\mathscr{X}$. The following theorem exploits this idea.

Theorem 3.1. *Let $u: \mathscr{R}^+ \times \mathscr{C} \to \mathscr{C}$ be a dynamical system on $\mathscr{C} \subset \mathscr{X}$, $\mathscr{X}$ a Banach space, and let $P: (\mathscr{C} \subset \mathscr{X}) \to \mathscr{X}$ be a compact operator with $\mathscr{R}(P) \subset \mathscr{C}$ and such that*

$$Pu(t, \phi) = u(t, Pu) \qquad \text{for } t \geqslant 0, \quad \phi \in \mathscr{C}. \tag{3.1}$$

*Then $\mathscr{R}(P)$ is positive invariant under $u$ and, if $\gamma^+(\phi)$ is bounded for some given $\phi \in \mathscr{C}$, then $\gamma^+(P\phi)$ is precompact in $\mathscr{X}$. Moreover, if $\mathscr{C} \subset \overline{\mathscr{R}(P)}$, all motions of $u$ are stable, and there exists an equilibrium, then all positive orbits are precompact.*

*Proof.* Given an arbitrary $\psi \in \mathscr{R}(P)$, $\psi = P\phi$ for some $\phi \in \mathscr{C}$. Thus

$$\gamma^+(\psi) = \bigcup_{t \geqslant 0} u(t, P\phi) = \bigcup_{t \geqslant 0} Pu(t, \phi) \subset \mathscr{R}(P)$$

and, hence, $\mathscr{R}(P)$ is positive invariant under $u$. Let $\phi \in \mathscr{C}$ be such that $\gamma^+(\phi)$ is bounded; hence $\gamma^+(P\phi) = P\gamma^+(\phi)$ is precompact. Now, if $\mathscr{C}$ contains an equilibrium and all motions are stable, then all motions are bounded and $\gamma^+(\psi)$ is precompact if $\psi \in \mathscr{R}(P)$. If $\mathscr{C} \subset \overline{\mathscr{R}(P)}$ it follows that the set of $\phi \in \mathscr{C}$ such that $\gamma^+(\phi)$ is precompact is a dense set in $\mathscr{C}$. Therefore, Proposition 3.4 of [13] provides the result that all positive orbits are precompact.

When the dynamical system is generated by (1.1) the following corollary is of interest.

Corollary 3.2. *If in Theorem* 3.1 *the dynamical system $u$ is generated by*

(1.1), $P\mathscr{D}(A) \subset \mathscr{D}(A)$, *and* $P$ *has a linear extension defined on all of* $\mathscr{X}$, *then condition* (3.1) *is equivalent to*

$$PAx = APx \qquad \textit{for } x \in \mathscr{D}(A). \tag{3.2}$$

*Proof.* Since $Pu(t, \cdot)$ and $u(t, P\cdot)$ are continuous mappings from $\mathscr{X}$ into $\mathscr{X}$, the proof is analogous to the latter part of the proof of Proposition 2.2.

If $\mathscr{X}$ is finite dimensional, the choice $P = I$ satisfies (3.1) and yields the obvious result that all bounded positive orbits are precompact. Similarly, if the semigroup $\{u(t, \cdot)\}_{t \geqslant 0}$ is compact [14], the choice $P = u(\tau, \cdot)$ satisfies (3.1) for any fixed $\tau > 0$ and provides the result (again transparent) that all bounded positive orbits are precompact.

Now consider the linear case in which $\mathscr{C} = \mathscr{X}$, $u$ is generated by (1.1), and $A$ is a linear operator with $\overline{\mathscr{D}(A)} = \mathscr{X}$. Then $u$ has an equilibrium at $x = 0$, stability of this equilibrium implies the stability and boundedness of all motions, and the following corollary may be useful.

COROLLARY 3.3. *Let* $u: \mathscr{R}^+ \times \mathscr{X} \to \mathscr{X}$ *be a linear dynamical system generated by* (1.1), *where* $A$ *is a linear operator with* $\overline{\mathscr{D}(A)} = \mathscr{X}$ *and* $\mathscr{X}$ *is a Banach space. Suppose there exists a polynomial* $p(A)$, $\mathscr{D}(p(A)) \subset \mathscr{D}(A)$, *such that either*

(i) $p(A)$ *is compact,* $\mathscr{D}(p(A)) = \mathscr{X}$, $\overline{\mathscr{R}(p(A))} = \mathscr{X}$, *or*

(ii) $(p(A))^{-1}$ *is compact,* $\overline{\mathscr{D}(p(A))} = \mathscr{X}$, $\mathscr{R}(p(A)) = \mathscr{X}$.

*Then if the equilibrium* $x = 0$ *is stable, all positive orbits are precompact.*

*Proof.* In case (i) let $P = p(A)$. Then $P: \mathscr{X} \to \mathscr{X}$, $P$ is linear, $\overline{\mathscr{R}(P)} = \mathscr{X}$, $P\mathscr{D}(A) \subset \mathscr{R}(P) \subset \mathscr{X} = \mathscr{D}(A)$, and $PAx = APx$ for $x \in \mathscr{D}(A) = \mathscr{X}$. Hence the condition of Corollary 3.2 and all conditions of Theorem 3.1 are satisfied.

In case (ii) define $P = (p(A))^{-1}$. Then $P: \mathscr{X} \to \mathscr{X}$, P is linear,

$$\overline{\mathscr{R}(P)} = \overline{\mathscr{D}(p(A))} = \mathscr{X},$$

$$P\mathscr{D}(A) \subset \mathscr{R}(P) = \mathscr{D}(p(A)) \subset \mathscr{D}(A),$$

and

$$PAx = PAp(A)\, Px = Pp(A)\, APx = APx \quad \text{for } x \in \mathscr{D}(A).$$

Again, the condition of Corollary 3.2 and all conditions of Theorem 3.1 are satisfied.

The last corollary provides a general explicit criterion for the linear case, but it is not as broad as Theorem 3.1. Indeed, if $\mathscr{X}$ is infinite-dimensional, $u$ is generated by (1.1), and $-A$ is the zero operator or the identity on $\mathscr{D}(A) = \mathscr{X}$, then Corollary 3.3 fails whereas Theorem 3.1 provides a result through the use of any linear compact operator $P: \mathscr{X} \to \mathscr{X}$ with $\overline{\mathscr{R}(P)} = \mathscr{X}$.

When the semigroup is not linear it may be very difficult to find any operator, other than the identity, which satisfies the commutivity conditions of Theorem 3.1. Since the identity is not compact unless $\mathscr{X}$ is finite dimensional, the following result is of interest.

THEOREM 3.4. *Let $u: \mathscr{R}^+ \times \mathscr{C} \to \mathscr{C}$ be a dynamical system on $\mathscr{C} \subset \mathscr{X}$, $\mathscr{X}$ a Banach space. Let there exist a sequence $\{P_n\}$ of compact operators, $P_n: (\mathscr{C} \subset \mathscr{X}) \to \mathscr{X}$, and a bounded operator $P: (\mathscr{C} \subset \mathscr{X}) \to (\mathscr{C} \subset \mathscr{X})$ such that*

$$Pu(t, \phi) = u(t, P\phi) \qquad \text{for } t \geqslant 0, \quad \phi \in \mathscr{C},$$

*and, for each $\psi$ in some subset $\mathscr{S}$ of $\mathscr{C}$,*

$$P_n u(t, \psi) \to Pu(t, \psi) \qquad \text{as} \qquad n \to \infty,$$

*uniformly in $t \geqslant 0$. Then $\gamma^+(P\phi)$ is precompact provided $\phi \in \mathscr{S}$ and $\gamma^+(\phi)$ is bounded. Moreover, if there exists an equilibrium, all motions of $u$ are stable, and $P\mathscr{S}$ is dense in $\mathscr{C}$, then all positive orbits are precompact.*

*Proof of Theorem* 3.4. Let $\phi \in \mathscr{S}$ be such that $\gamma^+(\phi)$ is bounded and let $\{t_m\}$ be an arbitrary increasing sequence with $t_m \to \infty$ as $m \to \infty$. Since $\{u(t_m, \phi)\}$ is bounded and all $P_n$ are compact, Cantor's diagonalization process yields a subsequence $\{t_{m'}\}$ such that $\{P_n u(t_{m'}, \phi)\}$ is Cauchy for every fixed $n$. It follows that for any given $\epsilon > 0$ there exist $n$, $N$ such that

$$\| P_n u(t, \phi) - Pu(t, \phi) \| < \epsilon/3 \qquad \text{for } t \geqslant 0,$$
$$\| P_n u(t_r, \phi) - P_n u(t_s, \phi) \| < \epsilon/3 \qquad \text{for } r, s > N,$$

where $t_r, t_s \in \{t_{m'}\}$. Since

$$\begin{aligned} &\| u(t_r, P\phi) - u(t_s, P\phi) \| \\ &\quad \leqslant \| u(t_r, P\phi) - P_n u(t_r, \phi) \| + \| P_n u(t_r, \phi) - P_n u(t_s, \phi) \| \\ &\qquad + \| P_n u(t_s, \phi) - u(t_s, P\phi) \|, \end{aligned}$$

the sequence $\{u(t_{m'}, P\phi)\}$ is Cauchy and therefore $\gamma^+(P\phi)$ is precompact. The remaining conclusions follow immediately from the proof of Theorem 3.1.

If one chooses $\mathscr{S} = \mathscr{C}$ and $P_n = P$ for all $n$, then this theorem yields Theorem 3.1 as a special case. The main advantage of this theorem as formulated is that $P$ need not be compact; in fact it might be possible to choose $P$

as the identity on $\mathscr{C}$ for a suitable choice of $\{P_n\}$ and $\mathscr{S} \subset \mathscr{C}$. Corollary 3.5 will illustrate this approach.

We will utilize the terminology of [15, 16] in order to obtain a result compatible with most recent results for semigroups of nonlinear contractions. Consider an accretive multivalued (set-valued) "operator" $B$: $\mathscr{D}(B) \to \mathscr{X}$, $\mathscr{D}(B) \subset \mathscr{X}$, $\mathscr{X}$ a Banach space. Then $-B$ is said to be the *generator* of a (generally nonlinear) contraction semigroup defined by

$$u(t, \phi) = \lim_{n\to\infty} \left(I + \frac{t}{n} B\right)^{-n} \phi \qquad \text{for } t \geqslant 0, \quad \phi \in \overline{\mathscr{D}(B)},$$

provided $\mathscr{R}(I + \lambda B) \supset \mathscr{D}(B)$ for all sufficiently small positive real $\lambda$ [15, 16]. It is known that $(I + \lambda B)^{-1}$ exists as a single-valued operator for all $\lambda > 0$.[2]

COROLLARY 3.5. *Let $B$ be a (generally multivalued) accretive "operator" on a Banach space $\mathscr{X}$ such that $0 \in \mathscr{R}(B)$, $\mathscr{R}(I + \lambda B) \supset \overline{\mathscr{D}(B)}$, and $(I + \lambda B)^{-m}$ is compact for all sufficiently small $\lambda > 0$ and some integer $m > 0$. Let $\{u(t, \cdot)\}_{t\geqslant 0}$ be the contraction semigroup on $\mathscr{C} = \overline{\mathscr{D}(B)}$ generated by $-B$. Then all positive orbits are precompact.*

*Proof.* Define $J_\lambda = (I + \lambda B)^{-1}$, $B_\lambda = \lambda^{-1}(I - J_\lambda)$,

$$\mathscr{S} = \{x \in \mathscr{C} \mid | Bx | \overset{\text{def}}{=} \sup_{\lambda>0} \| B_\lambda x \| < \infty\}$$

Crandall [16, Theorem 1 and Remark 2] shows that $\mathscr{D}(B) \subset \mathscr{S}$ (hence $\mathscr{S}$ is dense in $\mathscr{C}$), $\mathscr{S}$ is positive invariant under $u$, and

$$| B\phi | = \lim_{h\downarrow 0} \frac{1}{h} \| u(h, \phi) - \phi \| \qquad \phi \in \mathscr{S}.$$

Since $u(t, \cdot)$ is a contraction, this last result implies

$$| Bu(t, \phi)| \leqslant | B\phi | , \qquad \phi \in \mathscr{S}.$$

Since $0 \in \mathscr{R}(B)$ there exists $x_0 \in \mathscr{D}(B)$ such that $x_0 \in (I + \lambda B)\, x_0$. Hence $J_\lambda x_0 = x_0$, $u(t, x_0) = x_0$, and $x_0$ is an equilibrium. Since $u(t, \cdot)$ is a contraction all motions are stable. Hence all positive orbits are bounded.

[2] Under certain additional assumptions a semigroup with accretive generator $-B$ can be shown to be also generated by (1.1), where $A$ is a (single-valued) selection for $-B$ [15].

For $x \in \mathscr{C}$ define $Px = x$, $P_n x = J_{1/n}^m x$, for $n$ sufficiently large. Clearly, $P_n: (\mathscr{C} \subset \mathscr{X}) \to \mathscr{X}$ is compact and, for $\phi \in \mathscr{S}$,

$$\begin{aligned}
\| P_n u(t,\phi) - Pu(t,\phi)\| &= \| J_{1/n}^m u(t,\phi) - u(t,\phi)\| \\
&\leqslant \sum_{p=1}^{m} \| J_{1/n}^p u(t,\phi) - J_{1/n}^{p-1} u(t,\phi)\| \\
&\leqslant m \| J_{1/n} u(t,\phi) - u(t,\phi)\| \\
&= \frac{m}{n} \| B_{1/n} u(t,\phi)\| \leqslant \frac{m}{n} \mid Bu(t,\phi) \mid \leqslant \frac{m}{n} \mid B\phi \mid .
\end{aligned}$$

Since $P\mathscr{S} = \mathscr{S}$ is dense in $\mathscr{C}$, Theorem 3.4 states that all positive orbits are precompact.

We remark that when $B$ is linear it can be shown that $u$ is generated by (1.1) with $A = -B$, and then Corollary 3.5 is a special case of Corollary 3.3. Dafermos and Slemrod [6] also have obtained Corollary 3.5 for $m = 1$. Other results of a nature similar to Corollary 3.5 can be obtained from Theorem 3.4 through different choices of $P$ and $\{P_n\}$.

It is unfortunate that Corollary 3.5 applies only to contraction semigroups, but at present comparatively little is known concerning the properties of nonlinear noncontractive semigroups. One important property of a contraction semigroup is the existence of a set $\mathscr{S}$, dense in $\mathscr{C}$, such that for each $\phi \in \mathscr{S}$ the motion $u(\cdot,\phi)$ is uniformly Lipschitz continuous on $\mathscr{R}^+$ [16]. Since it appears that many noncontractive nonlinear semigroups may have a similar property, the following result may prove to be useful.

COROLLARY 3.6. *Let $u: \mathscr{R}^+ \times \mathscr{C} \to \mathscr{C}$ be a dynamical system on a closed set $\mathscr{C} \subset \mathscr{X}$, $\mathscr{X}$ a Banach space. Let $u$ be generated by* (1.1) *where $A = L + N$, $\mathscr{D}(A) = \mathscr{D}(N) = \mathscr{D}(L) \cap \mathscr{C}$, $N$ maps bounded sets into bounded sets, and $-L$ is accretive; moreover, assume that $\mathscr{R}(I - \lambda L) = \mathscr{X}$ and $(I - \lambda L)^{-m}$ is compact for all sufficiently small $\lambda > 0$ and some integer $m > 0$. Then $\gamma^+(\phi)$ is precompact if $\gamma^+(\phi)$ is bounded, $\phi \in \mathscr{D}(A)$, and $u(\cdot,\phi)$ is uniformly Lipschitz continuous on $\mathscr{R}^+$.*

*Proof.* Consider $\phi \in \mathscr{D}(A)$ such that $\gamma^+(\phi)$ is bounded and $u(\cdot,\phi)$ is uniformly Lipschitz continuous on $\mathscr{R}^+$. Thus there exists $M_\phi > 0$ such that $\| u(t+h,\phi) - u(t,\phi)\| \leqslant hM_\phi$ for all $t, h \in \mathscr{R}^+$. Thus

$$\| \dot{u}(t,\phi)\| = \| Au(t,\phi)\| \leqslant M_\phi \text{ a.e.} \qquad t \geqslant 0.$$

Define the operators $P_n = (I - (1/n)L)^{-m}$, $P = I$ on $\mathscr{C}$, and note that $P_n$

is compact for all sufficiently large $n$. Since $u(t, \phi) \in \mathscr{D}(A) = \mathscr{D}(L) \cap \mathscr{C}$ a.e. $t \geqslant 0$ and $-L$ is accretive [15],

$$\begin{aligned}
&\| P_n u(t, \phi) - Pu(t, \phi)\| \\
&\quad \leqslant \sum_{p=1}^{m} \|(I - (1/n) L)^{-p} u(t, \phi) - (I - (1/n) L)^{1-p} u(t, \phi)\| \\
&\quad \leqslant m \|(I - (1/n) L)^{-1} u(t, \phi) - u(t, \phi)\| \\
&\quad \leqslant (m/n) \| Lu(t, \phi)\| \\
&\quad \leqslant (m/n) M_\phi + (m/n) \| Nu(t, \phi)\| \quad \text{a.e.} \qquad t \geqslant 0.
\end{aligned}$$

Hence by the continuity of $P_n$, $P$, and $u(\cdot, \phi)$, the conditions of Theorem 3.4 are satisfied for $\mathscr{S} = \{\phi\}$.

Finally, we wish to mention that two interesting stability problems, previously published, could have been studied in a more direct manner through use of the present criteria. The results of Dafermos [17] for linear wave equations with weak damping could have been obtained from the invariance principle [3] using the total energy as a Liapunov functional, had it been known that all positive orbits were precompact. Precompactness of these orbits is clear since the evolution equations in [17] are such that $A$ has compact inverse and $p(A) = A$ satisfies condition (ii) of Corollary 3.3. In the linear thermoelastic problem considered by Slemrod and Infante [7] a similar remark applies, and the principal result stated there (Theorem 5.1 of [7]) is in fact valid for all initial data in

$$\mathscr{W}_{20}^{(1)}(\Omega) \times \mathscr{L}_2(\Omega) \times \mathscr{L}_2(\Omega).^3$$

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